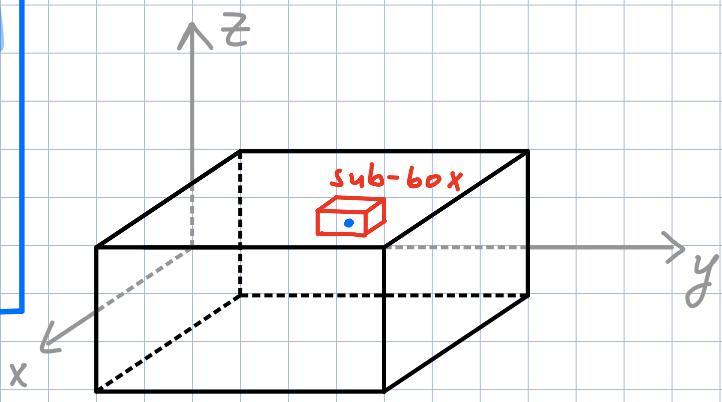
Last time:

Triple integrals

$$B = \left\{ (x,y,z) \,\middle|\, \begin{array}{l} a \le x \le b \\ c \le y \le d \\ r \le z \le s \end{array} \right\} \quad \text{rectangular box}$$



Triple integral: $f(x,y,z)$ - continuous fct

$$\iiint_B f(x,y,z)\, dV = \lim_{l,m,n} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\underbrace{x_{ijk}^*, y_{ijk}^*, z_{ijk}^*}_{\text{sample point in } (i\,j\,k)\text{-th sub-box}})\, \Delta V$$

Fubini's thm: if $f(x,y,z)$ continuous

$$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

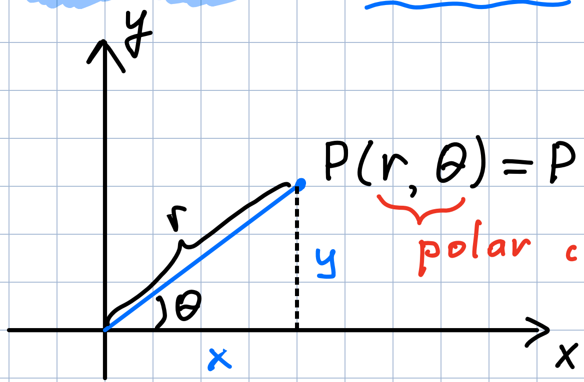
Rmk: Can be written in 5 other orders, e.g.

$$\int_c^d \int_a^b \int_r^s f(x,y,z)\, dz\, dx\, dy$$

Application:

$$\iiint_E 1\, dV = \text{Volume of the solid } E$$

Recall: Double integrals in polar coordinates



$P(r,\theta) = P(x,y)$

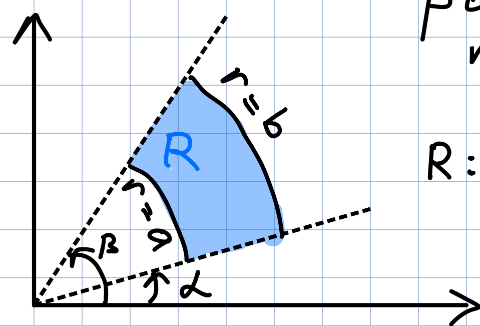
$\underbrace{r,\theta}_{\text{polar coordinates}}$

$x = r\cos\theta$

$y = r\sin\theta$

Assume: $0 \le a \le b$, $0 \le \beta - \alpha \le 2\pi$

polar rectangle

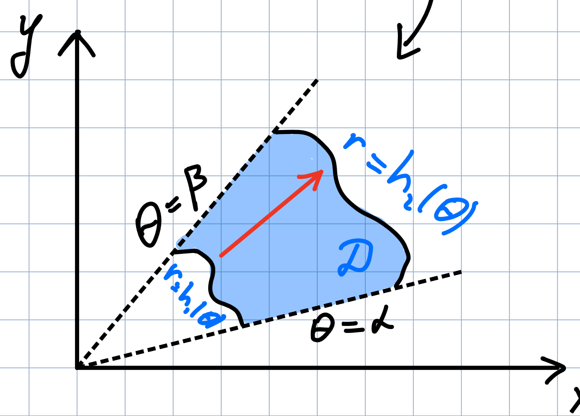


$R: \begin{array}{l} a \le r \le b \\ \alpha \le \theta \le \beta \end{array}$

don't forget this factor!

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,\underline{\underline{r}}\,dr\,d\theta$$

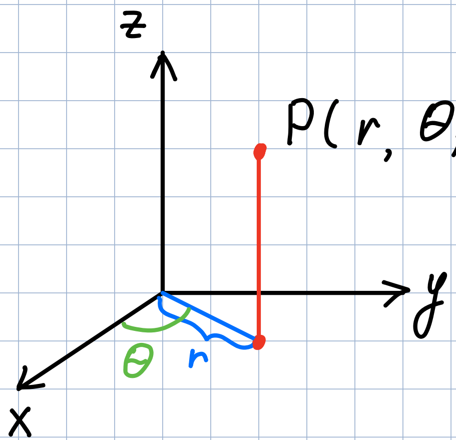
For D like that



$$D = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \end{array} \right\}$$

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Triple integrals in cylindrical coordinates



$\underbrace{r, \theta}_{\text{polar coord. of proj. of P onto xy-plane}}, z$ - cylindrical coordinates of a point P

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \Rightarrow x^2 + y^2 = r^2$$

$$z = z$$

For a solid

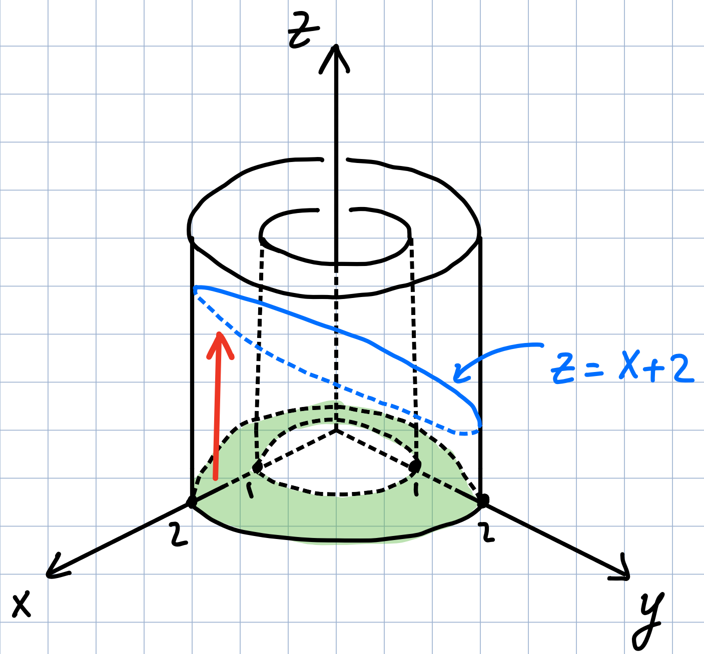
$$E: \left\{ \begin{array}{c} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \\ g_1(r,\theta) \le z \le g_2(r,\theta) \end{array} \right\}$$

$$\iiint_E f(x,y,z)\,dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, \underline{\underline{r}}\, dz\, dr\, d\theta$$

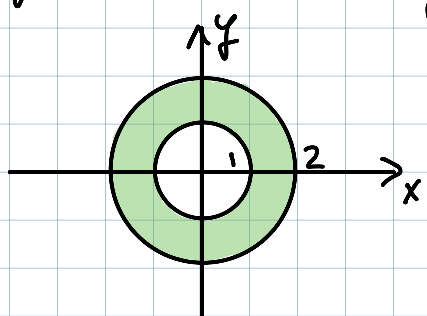
Ex: Compute $\iiint_E y\,dV$, where E – solid region below the plane $z=x+2$, above xy-plane and between cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

Sol:

Step 0: sketch E



Projection D to xy-plane:



Step 1: describe E in cylindrical coord.

$\dots \le \theta \le \dots, \quad \dots \le r \le \dots, \quad \dots \le z \le \dots$

in that order!

$0 \le \theta \le 2\pi$

$1 \le r \le 2$

$0 \le z \le \underbrace{r\cos\theta + 2}_{x+2}$

Step 3: $\iiint_E y\,dV$

$$= \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r\sin\theta \; r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r^2\sin\theta \; dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \underbrace{\int_0^{r\cos\theta+2} r^2 \sin\theta \, dz}_{z r^2 \sin\theta \Big|_{z=0}^{z=r\cos\theta+2} = (r\cos\theta+2) r^2 \sin\theta} \, dr \, d\theta = \int_0^{2\pi} \overbrace{\int_1^2 r^3 \sin\theta\cos\theta + 2r^2 \sin\theta \, dr}^{\frac{1}{4} r^4 \overbrace{\sin\theta\cos\theta}^{\frac{1}{2}\sin 2\theta} + \frac{2}{3} r^3 \sin\theta \Big|_{r=1}^{r=2}} \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{16-1}{4\cdot 2} \sin 2\theta + \frac{2}{3}(8-1)\sin\theta \right) d\theta = \int_0^{2\pi} \left(\frac{15}{8} \sin 2\theta + \frac{14}{3} \sin\theta \right) d\theta$$

$$= -\frac{15}{16} \cos 2\theta - \frac{14}{3} \cos\theta \Big|_{\theta=0}^{\theta=2\pi} = 0$$

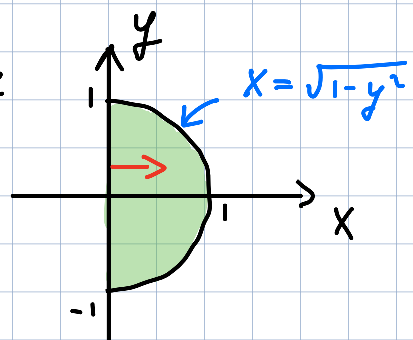
Ex: Convert to cylindrical

$$I = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dz\, dx\, dy \quad \text{and compute.}$$

Sol:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dz\, dx\, dy = \iiint_E xyz \, dV$$

Solid E above



with $x^2+y^2 \le z \le \sqrt{x^2+y^2}$

in cylindrical coordinates:

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le 1$$

$$r^2 \le z \le r$$

$$I = \underbrace{\int_{-\pi/2}^{\pi/2}\int_0^1 \int_{r^2}^{r} \underbrace{r\cos\theta \cdot r\sin\theta \cdot z}_{\frac{1}{2} z r^2 \sin 2\theta}\, r\, dz}_{\frac{1}{4} z^2 r^3 \sin 2\theta \Big|_{z=r^2}^{z=r}}\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \underbrace{\int_0^1 \frac{r^2-r^4}{4} r^3 \sin 2\theta \, dr}_{\frac{1}{4}\sin 2\theta \left(\frac{1}{6}r^6 - \frac{1}{8}r^8\right)\Big|_{r=0}^{r=1}}\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}\underbrace{\left(\frac{1}{6}-\frac{1}{8}\right)}_{\frac{1}{24}} \sin 2\theta \, d\theta = \frac{1}{96}\,\frac{-1}{2}\cos 2\theta \Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}$$